BONFERRONI SOMETIMES LOSES

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ABSTRACT

Bonferroni inequalities often provide tight upper and lower bounds for probabilities that are difficult to compute. However, there are situations in which the Bonferroni approach gives very poor results. An example is given in which the upper and lower Bonferroni bounds are far from the probability they seek to approximate and successive bounds do not converge.

KEY WORDS: Bonferroni inequalities; Runs distributions.

AUTHOR'S FOOTNOTE

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1. INTRODUCTION

The Bonferroni inequalities often provide an effective method for obtaining tight upper and lower bounds on probabilities that are difficult to compute. Their usefulness in a variety of important practical and theoretical situations is well known. It is not surprising to find the phrase "Bonferroni wins again!" in the title of a recent article (Bohrer et al., 1981) in which Bonferroni bounds performed better than several other methods of approximating hard-to-obtain critical values. Other examples of the utility of the Bonferroni technique include Angers and McLaughlin (1979), Bailey (1980), and Cook and Prescott (1981). A general treatment of Bonferroni inequalities appears in David (1981, Ch. 5).

For any events $A_1, \cdots, A_N$, the principle of inclusion and exclusion gives

$$\Pr\left[\bigcup_{i=1}^{N} A_i\right] = T_1 - T_2 + T_3 - T_4 \pm \cdots + (-1)^{N-1} T_N \quad , \tag{1.1}$$

where

$$T_1 = \sum_{i=1}^{N} \Pr[A_i], \qquad T_2 = \sum_{\substack{i,j=1 \\ i<j}}^{N} \Pr[A_i A_j], \qquad T_3 = \sum_{\substack{i,j,\ell=1 \\ i<j<\ell}}^{N} \Pr[A_i A_j A_\ell] \; , \; \cdots \; ,$$

and $T_N = \Pr[A_1 A_2 \cdots A_N]$. The sum of the first t terms on the right-hand side of (1.1) provides an upper bound to $\Pr[\cup A_i]$ when t is odd and a lower bound when t is even, producing the sequence of Bonferroni inequalities:

$$\begin{gathered} \Pr[\cup A_i] \le T_1, \qquad \Pr[\cup A_i] \le T_1 - T_2 + T_3, \; \cdots \\ T_1 - T_2 \le \Pr[\cup A_i], \qquad T_1 - T_2 + T_3 - T_4 \le \Pr[\cup A_i], \; \cdots \; . \end{gathered} \tag{1.2}$$

In many cases, these bounds lie close to $\Pr[\cup A_i]$ and increase in sharpness as the number of terms included from the right-hand side of (1.1) increases.

It is not widely realized that there are also situations in which the Bonferroni inequalities behave quite poorly. In particular, the closeness of upper and lower bounds to $\Pr[\cup A_i]$ and the convergence of successive bounds cannot be assumed. Neither of these properties holds when the Bonferroni approach is applied to the problem considered in this paper.

2. THE PROBLEM

Let $X_1, \cdots, X_n$ be a sequence of independent, identically distributed Bernoulli trials, that is, dichotomous random variables each taking the value 1 with probability p and 0 with probability $1-p$. Let A denote the event that this sequence of n trials contains a run of at least k successive 1's, where $k \geq 2$. Define $N \equiv n-k+1$. The event A occurs if and only if there is some i for which $1 \leq i \leq N$ and

$$X_i = X_{i+1} = X_{i+2} = \cdots = X_{i+k-1} = 1 \quad . \tag{2.1}$$

It is desired to find the probability that A occurs.

3. THE BONFERRONI APPROACH

Let A_i denote the event that there is a run of k 1's starting at trial i, i.e., that (2.1) holds, for $i = 1, \cdots, N$, so $A = \cup_{i=1}^{N} A_i$. The terms T_1 to T_4 of (1.1) will now be considered. Since $\Pr[A_i] = p^k$ for each i, it is immediate that

$$T_1 = \sum_{i=1}^{N} \Pr[A_i] = Np^k \, . \tag{3.1}$$

For $d = 1, \cdots, N-1$, let $S(d)$ denote the set of index pairs $\{(i,j)\colon 1 \leq i < j \leq N,\ j-i = d\}$. If $\#[d]$ denotes the number of pairs in $S(d)$ and $Q[d]$ denotes the common value of $\Pr[A_iA_j]$ for all pairs (i,j) in S(d), then $\#[d] = N-d$ and $Q[d] = p^{k+\min(d,k)}$. Summing the expression $\#[d]Q[d]$ over d gives

$$T_2 = \sum_{d=1}^{N-1} \sum_{S(d)} \Pr[A_i A_j] = \sum_{d=1}^{k-1} (N-d)p^{k+d} + \sum_{d=k}^{N-1} (N-d)p^{2k}$$

$$= Np^k G(0) - p^k G(1) + \tfrac{1}{2}(N-k)(N-k+1)p^{2k} \tag{3.2}$$

where

$$G(m) \equiv \sum_{h=1}^{k-1} h^m p^h \quad . \tag{3.3}$$

To calculate T_3, let $S(d,e)$ denote the set of index triples $\{(i,j,\ell): 1 \le i < j < \ell \le N,\ j-i = d,\ \ell - j = e\}$, where $1 \le d, e \le N-2$ and $d+e \le N-1$. The number of triples in this set is easily seen to be $\#[d,e] = N-d-e$, and for each of these triples,

$$Q[d,e] \equiv \Pr[A_i A_j A_\ell] = p^{k+\min(d,k)+\min(e,k)} \quad .$$

It is instructive to consider the relationship between the pair $[d,e]$ and the structure of the triples in $S(d,e)$. Summing $\Pr[A_i A_j A_\ell]$ over all (i,j,k) by summing $\#[d,e]Q[d,e]$ over all possible pairs $[d,e]$ gives

$$T_3 = \sum_{[d,e]} \sum_{S(d,e)} \Pr[A_i A_j A_\ell] = \sum_{[d,e]} \#[d,e]Q[d,e]$$

$$= \sum_{[d,e]} (N-d-e)p^{k+\min(d,k)+\min(e,k)} \quad . \tag{3.4}$$

To evaluate this sum, group the pairs $[d,e]$ by the exponent of p in $Q[d,e]$ as in Table 1, where

$$[h,k^*] \equiv \{[h,k],\ [h,k+1],\ \cdots,\ [h,N-h-1]\} \quad ,$$

$$[k^*,h] \equiv \{[k,h],\ [k+1,h],\ \cdots,\ [N-h-1,h]\} \quad ,$$

$$[k^*,k^*] \equiv [k,k^*] \cup [k+1,k^*] \cup [k+2,k^*] \cup \cdots \cup [N-k-1,k^*] \quad .$$

Thus $[h,k*]$ denotes the collection of all pairs $[d,e]$ for which the triples (i,j,ℓ) of $S(d,e)$ satisfy $j-i=h<k$, $\ell-j\geq k$, making $Q[d,e]=p^{2k+h}$; the collections $[k*,h]$ and $[k*,k*]$ may be described similarly. It is straightforward to find the number of distinct triples (i,j,ℓ) corresponding to each of these:

$$\#[h,k*]=\#[k*,h]=\sum_{e=k}^{N-h-1}\#[h,e]=\tfrac{1}{2}(N-k-h)(N-k-h+1)\quad,$$

$$\#[k*,k*]=\sum_{h=k}^{N-k-1}\#[h,k*]=\tfrac{1}{2}\sum_{i=1}^{N-2k}i(i+1)=(N-2k)(N-2k+1)(N-2k+2)/6\quad. \tag{3.5}$$

From the first segment of Table 1, there are $h-1$ patterns $[d,e]$ that result in an exponent of $k+h$ in (3.4), and for each of these $\#[d,e]=N-d-e=N-h$, where $2\leq h\leq k$. From the table's second segment, an exponent of $2k+h$ occurs with $k-h-1$ patterns $[d,e]$ for which $h<d,e<k$, each having $\#[d,e]=N-k-h$, and also from the collections $[h,k*]$ and $[k*,h]$. The third segment shows that the collection $[k*,k*]$ of patterns $[d,e]$ gives an exponent of $3k$. Using the exponent of p in $Q[d,e]$ as an index of summation in (3.4), with the aid of Table 1, the results of this paragraph, (3.5), and then (3.3), produces

$$\begin{aligned}T_3 &= \sum_{h=2}^{k}(h-1)(N-h)p^{k+h}+\sum_{h=1}^{k-1}[(N-k-h)(N-k-h+1)+(k-h-1)(N-k-h)]p^{2k+h}\\ &\qquad +(N-2k)(N-2k+1)(N-2k+2)p^{3k}/6\\ &= p^{k+1}[(N-1)G(1)-G(2)]+p^{2k}[N(N-k)G(0)-(3N-2k)G(1)+2G(2)]\\ &\qquad +(N-2k)(N-2k+1)(N-2k+2)p^{3k}/6\quad.\end{aligned}\tag{3.6}$$

Finally, using notation analogous to that just developed for T_3,

$$\begin{aligned}T_4 &= \sum_{[d,e,f]}\#[d,e,f]Q[d,e,f]\\ &= \sum_{[d,e,f]}(N-d-e-f)p^{k+\min(d,k)+\min(e,k)+\min(f,k)}\quad.\end{aligned}\tag{3.7}$$

Although T_4 could be computed by defining patterns $[d,e,k^*]$, etc., constructing a table like Table 1, and substituting into the formula just given, this lengthy calculation need not be fully treated. Instead, it suffices to obtain a lower bound L_4 for T_4 by summing the terms of equation (3.7) over all $[d,e,f]$ with $h \equiv d+e+f \leq k+1$. For any such $[d,e,f]$, the exponent of p in (3.7) is $k+h$, since $d,e,f < k$, and $N-d-e-f = N-h$, so

$$T_4 \geq L_4 = \sum_{\substack{[d,e,f] \\ h \equiv d+e+f \leq k+1}} (N-d-e-f)p^{k+\min(d,k)+\min(e,k)+\min(f,k)}$$

$$= \sum_{h=3}^{k+1} \mathcal{T}(h) \times (N-h)p^{k+h} \quad ,$$

where $\mathcal{T}(h)$ denotes the number of triples $[d,e,f]$ of positive integers with $d+e+f = h$. This is just the number of distinct ordered triples of non-negative integers whose three elements sum to $h-3$, or equivalently, the number of distinct arrangements of $h-3$ 0's and two 1's, so $\mathcal{T}(h) = \frac{1}{2}(h-1)(h-2)$, and

$$L_4 = \sum_{h=3}^{k+1} \tfrac{1}{2}(h-1)(h-2)(N-h)p^{k+h} = \tfrac{1}{2}p^{k+2}[(N-2)G(1) + (N-3)G(2) - G(3)] \quad . \qquad (3.8)$$

This derivation of L_4 has examined the first segment of a table analogous to Table 1, with the exponent of p ranging from $k+3$ to $2k+1$. For exponent $k+h$, the number of corresponding $[d,e,f]$ patterns is $\mathcal{T}(h)$. These patterns could be listed as in Table 1, e.g., for $h=6$, $\mathcal{T}(h)=10$, and the list consists of three permutations of $[1,1,4]$, six permutations of $[1,2,3]$, and $[2,2,2]$. The number of distinct sets of four event indices i,j,ℓ,m with $j-i = d$, $\ell - j = e$, $m - \ell = f$ is $\#[d,e,f] = N-h$, and p^{k+h} is $\Pr[A_i A_j A_\ell A_m]$ for any of these sets.

4. RESULTS AND CONCLUSIONS

The first three Bonferroni bounds of (1.2) and an upper limit for the fourth are obtained easily from equations (3.1), (3.2), (3.6), and (3.8). Tables 2 and 3 compare the first upper Bonferroni bound T_1, the first lower Bonferroni bound $T_1 - T_2$, the second upper Bonferroni bound $T_1 - T_2 + T_3$, and the upper limit $T_1 - T_2 + T_3 - L_4$ for the second lower Bonferroni bound to some exact values of Pr[A], whose derivation is discussed in the appendix. The ratios of these Bonferroni bounds to the corresponding exact probabilities appear in Table 3.

The first upper Bonferroni bound is larger than the exact probability by a factor ranging from 1.97 to 4.53, and the second upper Bonferroni bound is larger than the exact probability by a factor ranging from 1.92 to 39 . Neither of these bounds is close to the exact probability for any combination of n, k, and p, and the second upper bound is often greater than the first. The first lower Bonferroni bound is never more than .055 of the exact probability, and the upper limit for the second lower Bonferroni bound is never more than .111 of the exact probability. Both lower Bonferroni bounds are often negative, and the second lower bound is less than the first in many cases.

The bad behavior of these Bonferroni bounds is due to the dependent nature of the events A_i . Tight bounds result from equations (1.2) when some T_t is very small. (The exact value of $\Pr[\cup A_i]$ results when some T_t is zero.) The quantities $T_1, T_2, T_3, \cdots$ consist of n, $\binom{n}{2}$, $\binom{n}{3}$, $\cdots$ terms, each of which is a probability. Therefore T_2 cannot be small relative to T_1 unless all of its terms are small relative to terms in T_1 . This does not happen here, since

$$\Pr[A_iA_{i+1}] = p \Pr[A_i] \qquad \text{for } i = 1, \cdots, N-1$$

and

$$\Pr[A_iA_{i+2}] = p^2\Pr[A_i] \qquad \text{for } i = 1, \cdots, N-2 \quad ,$$

and so on. Summing these terms shows that T_2 is of the same order of magnitude as T_1. The same line of reasoning applies to T_3 and higher-order quantities.

The results of applying the Bonferroni approach to this problem indicate that the technique must be used with caution. The upper and lower bounds can be far from each other and from the value of $\Pr[\cup A_i]$. Successive bounds can fail to converge.

Bonferroni often wins — but it sometimes loses.

5. APPENDIX

To obtain exact values for Pr[A] of Section 2, define

$$f(m) \equiv \Pr[\text{the first run of } k \text{ 1's ends at trial } m] \quad ,$$

$$s(m) \equiv \Pr[\text{there is a run of } k \text{ 1's in the first } m \text{ trials}] \quad .$$

It is easy to establish the initial conditions

$$f(m) = \begin{cases} 0 & \text{for } m=1, \cdots, k-1 \\ p^k & \text{for } m=k \\ (1-p)p^k & \text{for } m=k+1, \cdots, 2k \quad , \end{cases}$$

$$s(m) = 0 \qquad \text{for } m=1, \cdots, k-1 \quad .$$

Two recursive relations provide $f(m)$ for $m>2k$ and $s(m)$ for all $m>k-1$. For the first run of k 1's to end at the mth trial where $m>2k$, there must be no run of k 1's in the first $m-k-1$ trials, which happens with probability $1-s(m-k-1)$, followed by a 0 and then by k successive 1's, so

$$f(m) = [1-s(m-k-1)](1-p)p^k \qquad \text{for } m=2k+1, \cdots, n \quad .$$

When a run of k 1's occurs in the first m trials, it must end either before or at trial m, so

$$s(m) = s(m-1) + f(m) \qquad \text{for } m = k, \cdots, n \quad .$$

These equations constitute a recursive solution to the problem. As m increases from k to n, f(m) and then s(m) can be computed for each m. At the end of this process, s(n) = Pr[A] has been obtained. This problem is treated in Feller (1968, Ch. XIII). A much more general version of it is solved in Schwager (1983).

REFERENCES

ANGERS, CLAUDE and McLAUGHLIN, GEORGES (1979), "A Note on Prediction Intervals for Large Future Samples," <u>Technometrics</u>, 21, 383-385.

BAILEY, B. J. R. (1980), "Large Sample Simultaneous Confidence Intervals for the Multinomial Probabilities Based on Transformations of the Cell Frequencies," <u>Technometrics</u>, 22, 583-589.

BOHRER, ROBERT, CHOW, WINSTON, FAITH, RAY, JOSHI, V. M., and WU, C.-F. (1981), "Multiple Three-Decision Rules for Factorial Simple Effects: Bonferroni Wins Again!", <u>Journal of the American Statistical Association</u>, 76, 119-124.

COOK, R. DENNIS and PRESCOTT, P. (1981), "On the Accuracy of Bonferroni Significance Levels for Detecting Outliers in Linear Models," <u>Technometrics</u>, 23, 59-63.

DAVID, HERBERT A. (1981), <u>Order Statistics</u>, <u>2nd</u> edition, New York: John Wiley and Sons, Inc.

FELLER, WILLIAM (1968), <u>An Introduction to Probability Theory and Its Applications</u>, <u>Vol. I</u>, 3<u>rd</u> edition, New York: John Wiley and Sons, Inc.

SCHWAGER, STEVEN J. (1983), "Run Probabilities in Sequences of Markov-Dependent Trials," to appear in <u>Journal of the American Statistical Association</u>.

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Table 2. First Three Bonferroni Bounds and an Upper Limit for the Fourth Bonferroni Bound to P[run of length k] for $p = 0.5(0.1)0.7$, $k = 15(5)30$, $n = 100(100)500$

Table 3. Ratios of First Three Bonferroni Bounds and an Upper Limit for the Fourth to P[run of length k] for $p = 0.5(0.1)0.7$, $k = 15(5)30$, $n = 100(100)500$

Table 1. Exponents of p in Q[d,e] and Corresponding [d,e] Patterns

Exponent of p	Corresponding [d,e] Patterns
k+2	[1,1]
k+3	[1,2] [2,1]
k+4	[1,3] [2,2] [3,1]
k+5	[1,4] [2,3] [3,2] [4,1]
⋮	⋮
k+h	[1,h-1] [2,h-2] ··· [h-1,1]
⋮	⋮
2k	[1,k-1] [2,k-2] ··· [k-1,1]
2k+1	[1,k*] [k*,1] [2,k-1] [3,k-2] ··· [k-1,2]
2k+2	[2,k*] [k*,2] [3,k-1] [4,k-2] ··· [k-1,3]
⋮	⋮
2k+h	[h,k*] [k*,h] [h+1,k-1] [h+2,k-2] ··· [k-1,h+1]
⋮	⋮
3k-1	[k-1,k*] [k*,k-1]
3k	[k*,k*]

Table 2. First Three Bonferroni Bounds and an Upper Limit for the Fourth Bonferroni Bound to P[run of length k] for p = 0.5(0.1)0.7, k = 15(5)30, n = 100(100)500

N	K	P	T1-T2+T3-L4	T1-T2	PR(K RUN)	T1	T1-T2+T3
100	15	0.5	0.1284E-03	0.5878E-04	0.1327E-02	0.2625E-02	0.2565E-02
200	15	0.5	0.1455E-03	0.4765E-04	0.2850E-02	0.5676E-02	0.5628E-02
300	15	0.5	0.1720E-03	0.2721E-04	0.4371E-02	0.8728E-02	0.8700E-02
400	15	0.5	0.2079E-03	-0.2548E-05	0.5889E-02	0.1178E-01	0.1178E-01
500	15	0.5	0.2532E-03	-0.4162E-04	0.7405E-02	0.1483E-01	0.1487E-01
100	20	0.5	0.3822E-05	0.1906E-05	0.3910E-04	0.7725E-04	0.7534E-04
200	20	0.5	0.3842E-05	0.1896E-05	0.8678E-04	0.1726E-03	0.1707E-03
300	20	0.5	0.3871E-05	0.1877E-05	0.1345E-03	0.2680E-03	0.2661E-03
400	20	0.5	0.3910E-05	0.1848E-05	0.1821E-03	0.3634E-03	0.3615E-03
500	20	0.5	0.3957E-05	0.1811E-05	0.2298E-03	0.4587E-03	0.4569E-03
100	25	0.5	0.1192E-06	0.5960E-07	0.1147E-05	0.2265E-05	0.2205E-05
200	25	0.5	0.1192E-06	0.5959E-07	0.2638E-05	0.5245E-05	0.5186E-05
300	25	0.5	0.1193E-06	0.5957E-07	0.4128E-05	0.8225E-05	0.8166E-05
400	25	0.5	0.1193E-06	0.5955E-07	0.5618E-05	0.1121E-04	0.1115E-04
500	25	0.5	0.1194E-06	0.5951E-07	0.7108E-05	0.1419E-04	0.1413E-04
100	30	0.5	0.3725E-08	0.1863E-08	0.3353E-07	0.6612E-07	0.6426E-07
200	30	0.5	0.3725E-08	0.1863E-08	0.8009E-07	0.1593E-06	0.1574E-06
300	30	0.5	0.3725E-08	0.1863E-08	0.1267E-06	0.2524E-06	0.2505E-06
400	30	0.5	0.3725E-08	0.1862E-08	0.1732E-06	0.3455E-06	0.3437E-06
500	30	0.5	0.3725E-08	0.1862E-08	0.2198E-06	0.4387E-06	0.4368E-06
100	15	0.6	-0.5444E-01	-0.1898E-01	0.1637E-01	0.4044E-01	0.6818E-01
200	15	0.6	-0.1228E+00	-0.4512E-01	0.3475E-01	0.8745E-01	0.1556E+00
300	15	0.6	-0.1865E+00	-0.7347E-01	0.5278E-01	0.1345E+00	0.2476E+00
400	15	0.6	-0.2456E+00	-0.1040E+00	0.7048E-01	0.1815E+00	0.3443E+00
500	15	0.6	-0.2999E+00	-0.1368E+00	0.8785E-01	0.2285E+00	0.4458E+00
100	20	0.6	-0.4140E-02	-0.1346E-02	0.1206E-02	0.2962E-02	0.4913E-02
200	20	0.6	-0.1002E-01	-0.3189E-02	0.2666E-02	0.6618E-02	0.1134E-01
300	20	0.6	-0.1588E-01	-0.5045E-02	0.4124E-02	0.1027E-01	0.1779E-01
400	20	0.6	-0.2171E-01	-0.6914E-02	0.5580E-02	0.1393E-01	0.2427E-01
500	20	0.6	-0.2751E-01	-0.8797E-02	0.7034E-02	0.1759E-01	0.3078E-01
100	25	0.6	-0.3003E-03	-0.9738E-04	0.8813E-04	0.2161E-03	0.3568E-03
200	25	0.6	-0.7619E-03	-0.2396E-03	0.2018E-03	0.5004E-03	0.8545E-03
300	25	0.6	-0.1223E-02	-0.3819E-03	0.3155E-03	0.7847E-03	0.1352E-02
400	25	0.6	-0.1685E-02	-0.5243E-03	0.4292E-03	0.1069E-02	0.1850E-02
500	25	0.6	-0.2146E-02	-0.6668E-03	0.5429E-03	0.1353E-02	0.2349E-02
100	30	0.6	-0.2157E-04	-0.7019E-05	0.6411E-05	0.1570E-04	0.2581E-04
200	30	0.6	-0.5749E-04	-0.1807E-04	0.1525E-04	0.3780E-04	0.6450E-04
300	30	0.6	-0.9341E-04	-0.2913E-04	0.2410E-04	0.5991E-04	0.1032E-03
400	30	0.6	-0.1293E-03	-0.4018E-04	0.3294E-04	0.8202E-04	0.1419E-03
500	30	0.6	-0.1652E-03	-0.5124E-04	0.4178E-04	0.1041E-03	0.1806E-03
100	15	0.7	-0.2475E+01	-0.5599E+00	0.1205E+00	0.4083E+00	0.1715E+01
200	15	0.7	-0.4955E+01	-0.1459E+01	0.2398E+00	0.8830E+00	0.4667E+01
300	15	0.7	-0.6448E+01	-0.2584E+01	0.3428E+00	0.1358E+01	0.8606E+01
400	15	0.7	-0.6847E+01	-0.3934E+01	0.4319E+00	0.1833E+01	0.1364E+02
500	15	0.7	-0.6044E+01	-0.5510E+01	0.5089E+00	0.2307E+01	0.1987E+02
100	20	0.7	-0.4580E+00	-0.8105E-01	0.1984E-01	0.6463E-01	0.2464E+00
200	20	0.7	-0.1092E+01	-0.1943E+00	0.4313E-01	0.1444E+00	0.5989E+00
300	20	0.7	-0.1701E+01	-0.3140E+00	0.6588E-01	0.2242E+00	0.9754E+00
400	20	0.7	-0.2286E+01	-0.4400E+00	0.8808E-01	0.3040E+00	0.1376E+01
500	20	0.7	-0.2846E+01	-0.5724E+00	0.1098E+00	0.3838E+00	0.1803E+01
100	25	0.7	-0.7377E-01	-0.1257E-01	0.3149E-02	0.1019E-01	0.3814E-01
200	25	0.7	-0.1872E+00	-0.3063E-01	0.7156E-02	0.2360E-01	0.9389E-01
300	25	0.7	-0.3000E+00	-0.4886E-01	0.1115E-01	0.3701E-01	0.1503E+00
400	25	0.7	-0.4122E+00	-0.6728E-01	0.1512E-01	0.5042E-01	0.2074E+00
500	25	0.7	-0.5236E+00	-0.8588E-01	0.1908E-01	0.6384E-01	0.2651E+00
100	30	0.7	-0.1151E-01	-0.1959E-02	0.4958E-03	0.1600E-02	0.5937E-02
200	30	0.7	-0.3082E-01	-0.4969E-02	0.1172E-02	0.3854E-02	0.1522E-01
300	30	0.7	-0.5011E-01	-0.7983E-02	0.1847E-02	0.6108E-02	0.2452E-01
400	30	0.7	-0.6938E-01	-0.1100E-01	0.2522E-02	0.8362E-02	0.3384E-01
500	30	0.7	-0.8863E-01	-0.1403E-01	0.3196E-02	0.1062E-01	0.4318E-01

Table 3. Ratios of First Three Bonferroni Bounds and an Upper Limit for the Fourth to P[run of length k] for $p = 0.5(0.1)0.7$, $k = 15(5)30$, $n = 100(100)500$

N	K	P	LOWER2/PR	LOWER1/PR	UPPER1/PR	UPPER2/PR
100	15	0.5	0.09676	0.04430	1.97790	1.93341
200	15	0.5	0.05107	0.01672	1.99169	1.97476
300	15	0.5	0.03936	0.00623	1.99694	1.99050
400	15	0.5	0.03531	-0.00043	2.00027	2.00050
500	15	0.5	0.03420	-0.00562	2.00285	2.00830
100	20	0.5	0.09775	0.04874	1.97563	1.92689
200	20	0.5	0.04427	0.02184	1.98908	1.96723
300	20	0.5	0.02879	0.01396	1.99302	1.97906
400	20	0.5	0.02147	0.01015	1.99493	1.98477
500	20	0.5	0.01722	0.00788	1.99606	1.98817
100	25	0.5	0.10390	0.05195	1.97403	1.92208
200	25	0.5	0.04521	0.02259	1.98870	1.96611
300	25	0.5	0.02889	0.01443	1.99278	1.97835
400	25	0.5	0.02124	0.01060	1.99470	1.98410
500	25	0.5	0.01679	0.00837	1.99581	1.98744
100	30	0.5	0.11111	0.05555	1.97222	1.91667
200	30	0.5	0.04651	0.02325	1.98837	1.96512
300	30	0.5	0.02941	0.01470	1.99265	1.97794
400	30	0.5	0.02150	0.01075	1.99462	1.98387
500	30	0.5	0.01695	0.00847	1.99576	1.98729
100	15	0.6	-3.32652	-1.15988	2.47087	4.16626
200	15	0.6	-3.53374	-1.29865	2.51704	4.47886
300	15	0.6	-3.53423	-1.39201	2.54773	4.69169
400	15	0.6	-3.48492	-1.47606	2.57505	4.88550
500	15	0.6	-3.41391	-1.55728	2.60115	5.07482
100	20	0.6	-3.43277	-1.11595	2.45538	4.07296
200	20	0.6	-3.75972	-1.19595	2.48206	4.25286
300	20	0.6	-3.85078	-1.22320	2.49115	4.31417
400	20	0.6	-3.89096	-1.23909	2.49644	4.34993
500	20	0.6	-3.91186	-1.25068	2.50030	4.37603
100	25	0.6	-3.40773	-1.10497	2.45166	4.04866
200	25	0.6	-3.77479	-1.18715	2.47906	4.23356
300	25	0.6	-3.87696	-1.21041	2.48681	4.28589
400	25	0.6	-3.92474	-1.22158	2.49053	4.31100
500	25	0.6	-3.95231	-1.22824	2.49275	4.32600
100	30	0.6	-3.36410	-1.09484	2.44828	4.02588
200	30	0.6	-3.76879	-1.18482	2.47827	4.22835
300	30	0.6	-3.87643	-1.20879	2.48626	4.28227
400	30	0.6	-3.92626	-1.21990	2.48997	4.30728
500	30	0.6	-3.95498	-1.22633	2.49211	4.32174
100	15	0.7	-20.53670	-4.64552	3.38752	14.22799
200	15	0.7	-20.66898	-6.08652	3.68314	19.46402
300	15	0.7	-18.80995	-7.53759	3.96073	25.10262
400	15	0.7	-15.85280	-9.10878	4.24294	31.57835
500	15	0.7	-11.87648	-10.82620	4.53374	39.05046
100	20	0.7	-23.08786	-4.08610	3.25841	12.42060
200	20	0.7	-25.30812	-4.50529	3.34835	13.88432
300	20	0.7	-25.82460	-4.76603	3.40358	14.80658
400	20	0.7	-25.95738	-4.99524	3.45152	15.62769
500	20	0.7	-25.93369	-5.21485	3.49688	16.42390
100	25	0.7	-23.42509	-3.99059	3.23636	12.11096
200	25	0.7	-26.16542	-4.27971	3.29840	13.12125
300	25	0.7	-26.91805	-4.38371	3.32069	13.48507
400	25	0.7	-27.25882	-4.44947	3.33477	13.71551
500	25	0.7	-27.44576	-4.50105	3.34578	13.89659
100	30	0.7	-23.21953	-3.95059	3.22755	11.97460
200	30	0.7	-26.30577	-4.24087	3.28977	12.99026
300	30	0.7	-27.13075	-4.32261	3.30729	13.27623
400	30	0.7	-27.51154	-4.36338	3.31603	13.41889
500	30	0.7	-27.72966	-4.38915	3.32155	13.50909